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On Waveguide Excitation By Source Placed On The Lateral Cross-Section

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ABSTRACT

The problem of excitation of electromagnetic oscillations in a waveguide with metal walls, which has an arbitrary cross-section, is reduced to an infinite set of boundary-value problems for telegraph equations in a quarter of the plane. The values of the longitudinal components of the field or of the lateral components of the magnetic vector (surface currents) on the cross-section of the waveguide can be chosen as the wave sources.

It is preliminary shown that the components of the non-harmonic electromagnetic field in the waveguide are expanded into series by two sets of eigen functions of the two-dimensional Laplacian that satisfy the Neumann or Dirichlet boundary conditions. The coefficients of these expansions are the solutions of telegraph equations or derivatives of these solutions. The boundary-value problem for the telegraph equation in a quarter of the plane is considered. It has been established which boundary conditions are sufficient for determining its unique solution. The solvability conditions of the auxiliary over-determined boundary-value problem have been written down. The formulas that give an explicit solution of the telegraph equation in a quarter of the plane in the case of different boundary conditions have been obtained. It is shown how to determine the boundary values of the solutions of the telegraph equations for various types of sources of the electromagnetic field.

As an example, the longitudinal components of the field for the high mode of a rectangular waveguide for given pulse sources are determined numerically.

Keywords: metal waveguide, wave excitation, telegraph equation, cross-sectional source.

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1. INTRODUCTION

The basic research stages in studying the waveguide structure include both the description of the set of eigen waves and the search for the conditions for their excitation (Barybin,2007).

In the case of a harmonic time-dependent electromagnetic field, the foundations of the theory of waveguides with metal walls were created in the middle of the last century (see, for example, the works by A. A. Samarskii and A. N. Tikhonov (Samarskii,1948; Samarskii,1948 in Russian). The problem of field excitation by the currents defined inside the waveguide was investigated in sufficient details. The modern theory of waveguides excitation of various types is presented in the review article (Solncev,2009).

For metal waveguides, the cases are known where the solutions of the problems of propagation and diffraction of natural waves can be solved analytically (Collin,1960, Mittra,1971). In this paper we use explicit solutions of boundary-value problems for a telegraph equation in a quarter of the plane.

Metal waveguides have wide application in engineering. The excitation of oscillating processes with given characteristics in such structures is one of the problems facing the engineers. The waveguide modes, usually, excite with a post inside the waveguide (Kong,2000, Pan,2014). The post can be of both a simple dipole shape and a loop shape. Also, the eigen waves are excited through the waveguide slots or through another conjugate waveguide (Sadiku,2013).

In mathematical modeling of the process of excitation and propagation of waves in the waveguide, the limiting values of the components of the electromagnetic field on the waveguide walls (Sadiku,2013) or at its cross-sections can be considered as their sources. This paper concerns the problem of excitation of a non-harmonic field from the traces of vectors of electric and magnetic strength at the cross-section of the waveguide.

In this paper we have previously studied the structure of the non-harmonic electromagnetic field in metal waveguides. As it is known, a harmonic time-dependent electromagnetic field in waveguides with metal walls can be represented as a sum of TE- and TM-fields . It is shown that in the non-harmonic case the components of the field also expand into series the terms of which contain the solutions of telegraph equations or

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derivatives of these solutions as multipliers. A mixed boundary-value problem for the telegraph equation in a quarter of the plane bounded by two coordinate semi-axes has been analyzed. It is established which boundary conditions are sufficient to determine the unique solution of the telegraph equation.

It is shown that the problem of the waveguide field definition by sources in the cross-section is reduced to an infinite set of boundary problems for telegraph equations in a quadrant.

2. ELECTROMAGNETIC WAVES IN METAL WAVEGUIDE

Let axis z be the longitudinal axis of a cylindrical waveguide and its lateral section Ω be bounded sectionally by smooth contour C that is described by equations $x = x(\tau)$, $y = y(\tau)$. In Ω there exist two orthonormal sets $\psi_m(x, y)$, $\varphi_m(x, y)$, $m = 0, 1, \ldots$, of eigen-functions of the Laplace operator that satisfy different boundary conditions:

$$\Delta\psi_m + \chi_m\psi_m = 0, \quad \psi_m = 0 \text{ at } C,$$

$$\Delta\varphi_m + \lambda_m\varphi_m = 0, \quad \frac{\partial\varphi_m}{\partial\nu} = 0 \text{ at } C.$$

As it is known, in the case of time-harmonic electromagnetic field, each function $\psi_m(x, y)$ or $\varphi_m(x, y)$ is corresponded to an eigen wave of the metal waveguide with cross-section Ω. An analogous assertion exists in the case of an arbitrary dependence of the field component on time.

It is sufficient to specify the condition on metal walls: the tangential components of the vector E are equal to zero. But, as it is known, the normal component of the vector H in this case is also zero. We'll use a complete set of necessary boundary conditions for the Maxwell equations

$$E_z = 0, \quad x'(\tau)E_x + y'(\tau)E_y = 0, \tag{1}$$

$$-y'(\tau)H_x + x'(\tau)H_y = 0. \tag{2}$$

The components of the general solution of the Maxwell equations satisfying the conditions (1), (2) on the boundary of a cylindrical domain have the form

$$E_z = \sum_m E_{z,m}(z,t)\chi_m\psi_m(x,y), \tag{3}$$

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$$H_z = \sum_m H_{z,m}(z,t)\lambda_m \varphi_m(x,y), \tag{4}$$

$$H_y = -\varepsilon_0\varepsilon \sum_m \frac{\partial E_{z,m}(z,t)}{\partial t}\frac{\partial \psi_m}{\partial x}(x,y) + P, \tag{5}$$

$$P = \sum_m \frac{\partial H_{z,m}}{\partial z}\frac{\partial \varphi_m}{\partial y}(x,y),$$

$$H_x = \varepsilon_0\varepsilon \sum_m \frac{\partial E_{z,m}(z,t)}{\partial t}\frac{\partial \psi_m}{\partial y}(x,y) + Q, \tag{6}$$

$$Q = \sum_m \frac{\partial H_{z,m}}{\partial z}\frac{\partial \varphi_m}{\partial x}(x,y),$$

where $H_{z,m}(z, t)$ and $E_{z,m}(z, t)$ are the solutions of telegraph equations

$$\frac{\partial^2 H_{z,m}}{\partial t^2} = \frac{1}{\kappa^2}\frac{\partial^2 H_{z,m}}{\partial z^2} - \frac{\lambda_m}{\kappa^2} H_{z,m}, \tag{7}$$

$$\frac{\partial^2 E_{z,m}}{\partial t^2} = \frac{1}{\kappa^2}\frac{\partial^2 E_{z,m}}{\partial z^2} - \frac{\chi_m}{\kappa^2} E_{z,m} \tag{8}$$

$(\kappa^2 = \mu_0\mu\varepsilon_0\varepsilon, \quad m = 0,1,\ldots)$.

The proof can be carried out by analogy with the arguments of the paper (Solncev,2009). The Maxwell equations in a homogeneous isotropic medium without sources have the form

$$
\begin{array}{llll}
1) & \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} = \varepsilon_0\varepsilon\dfrac{\partial E_x}{\partial t}, & 4) & \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} = -\mu_0\mu\dfrac{\partial H_x}{\partial t}, \\
2) & \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} = \varepsilon_0\varepsilon\dfrac{\partial E_y}{\partial t}, & 5) & \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = -\mu_0\mu\dfrac{\partial H_y}{\partial t}, \\
3) & \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = \varepsilon_0\varepsilon\dfrac{\partial E_z}{\partial t}, & 6) & \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = -\mu_0\mu\dfrac{\partial H_z}{\partial t}.
\end{array}
$$

We'll seek the function E_z in the form (3), where the factor χ_m is used for convenience. We consider the third Maxwell equation. Suppose that the tangential components of the vector E have the forms (5) and (6).

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It is easy to see, that the functions $P(x, y, z, t)$ and $Q(x, y, z, t)$ should satisfy the condition

$$\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}=0.$$

Consequently,

$$P=\frac{\partial R}{\partial y},Q=\frac{\partial R}{\partial x},$$

where $R(x, y, z, t)$ is a certain (differentiable) function. Then the boundary condition (2) has the form

$$\varepsilon_0\varepsilon\sum_m \frac{\partial E_{z,m}}{\partial t}\left[-y'(\tau)\frac{\partial \psi_m}{\partial y}-x'(\tau)\frac{\partial \psi_m}{\partial x}\right]-y'(\tau)\frac{\partial R}{\partial x}+x'(\tau)\frac{\partial R}{\partial y}=0.$$

If $\psi_m(x(\tau), y(\tau)) = 0$, then

$$y'(\tau)\frac{\partial \psi_m}{\partial y}+x'(\tau)\frac{\partial \psi_m}{\partial x}=0.$$

The last component is $\partial R/\partial v$. Therefore,

$$R(x,y,z,t)=\sum_m R_m(z,t)\varphi_m(x,y)$$

and then

$$P(x,y,z,t)=\sum_m R_m(z,t)\frac{\partial \varphi_m}{\partial y}(x,y),\quad Q(x,y,z,t)=\sum_m R_m(z,t)\frac{\partial \varphi_m}{\partial x}(x,y).$$

It follows from the fourth and fifth equations of the Maxwell system that

$$\frac{\partial E_y}{\partial z}=\sum_m \frac{\partial \psi_m}{\partial y}\left(\chi_m E_{z,m}+\kappa^2\frac{\partial^2 E_{z,m}}{\partial t^2}\right)+\mu_0\mu\sum_m \frac{\partial R_m}{\partial t}\frac{\partial \varphi_m}{\partial x},$$

$$\frac{\partial E_x}{\partial z}=\sum_m \frac{\partial \psi_m}{\partial x}\left(\chi_m E_{z,m}+\kappa^2\frac{\partial^2 E_{z,m}}{\partial t^2}\right)-\mu_0\mu\sum_m \frac{\partial R_m}{\partial t}\frac{\partial \varphi_m}{\partial y}.$$

We differentiate the sixth equation by z and substitute these expressions into it. We get

$$-\mu_0\mu\sum_m \frac{\partial^2 H_z}{\partial z\partial t}=\frac{\partial^2 E_y}{\partial z\partial x}-\frac{\partial^2 E_x}{\partial z\partial y}=\mu_0\mu\sum_m\left(\frac{\partial^2\varphi_m}{\partial x^2}+\frac{\partial^2\varphi_m}{\partial y^2}\right)\frac{\partial R_m}{\partial t}=-\mu_0\mu\sum_m \lambda_m\varphi_m\frac{\partial R_m}{\partial t}.$$

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Therefore, we have formula (4), where

$$\frac{\partial H_{z,m}}{\partial z} = R_m .$$

Now

$$\frac{\partial E_y}{\partial z} = \sum_m \frac{\partial \psi_m}{\partial y}\left(\chi_m E_{z,m} + \kappa^2 \frac{\partial^2 E_{z,m}}{\partial t^2}\right) + \mu_0 \mu \sum_m \frac{\partial \varphi_m}{\partial x}\frac{\partial^2 H_{z,m}}{\partial z \partial t},$$

$$\frac{\partial E_x}{\partial z} = \sum_m \frac{\partial \psi_m}{\partial x}\left(\chi_m E_{z,m} + \kappa^2 \frac{\partial^2 E_{z,m}}{\partial t^2}\right) - \mu_0 \mu \sum_m \frac{\partial \varphi_m}{\partial y}\frac{\partial H_{z,m}}{\partial z \partial t}.$$

Consequently,

$$E_y = \sum_m \frac{\partial \psi_m}{\partial y} S_m(z,t) + \mu_0 \mu \sum_m \frac{\partial \varphi_m}{\partial x}\frac{\partial H_{z,m}}{\partial t},$$

$$E_x = \sum_m \frac{\partial \psi_m}{\partial x} S_m(z,t) - \mu_0 \mu \sum_m \frac{\partial \varphi_m}{\partial y}\frac{\partial H_{z,m}}{\partial t},$$

where the functions $S_m(z, t)$ are to satisfy the condition

$$\frac{\partial S_m}{\partial z} = \chi_m E_{z,m} + \kappa^2 \frac{\partial^2 E_{z,m}}{\partial t^2}.$$

It is easy to verify that the second boundary condition in (1) is satisfied.

We substitute the expansions of the unknown functions into the first and the second Maxwell equations and obtain

$$\sum_m \frac{\partial \varphi_m}{\partial y}\left[\lambda_m H_{z,m} + \kappa^2 \frac{\partial^2 H_{z,m}}{\partial t^2} - \frac{\partial R_m}{\partial z}\right] + \varepsilon_0 \varepsilon \sum_m \frac{\partial \psi_m}{\partial x}\left[\frac{\partial^2 E_{z,m}}{\partial z \partial t} - \frac{\partial S_m}{\partial t}\right] = 0,$$

$$-\sum_m \frac{\partial \varphi_m}{\partial x}\left[\lambda_m H_{z,m} + \kappa^2 \frac{\partial^2 H_{z,m}}{\partial t^2} - \frac{\partial R_m}{\partial z}\right] + \varepsilon_0 \varepsilon \sum_m \frac{\partial \psi_m}{\partial y}\left[\frac{\partial^2 E_{z,m}}{\partial z \partial t} - \frac{\partial S_m}{\partial t}\right] = 0.$$

Consequently,

$$\lambda_m H_{z,m} + \kappa^2 \frac{\partial^2 H_{z,m}}{\partial t^2} - \frac{\partial R_m}{\partial z} = 0, \quad \frac{\partial E_{z,m}}{\partial z} - S_m = 0.$$

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As a result, we have the sets of telegraph equations (7) and (8).

3. TELEGRAPH EQUATION IN THE QUARTER OF THE PLANE

Consider the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial z^2} - b^2 u \qquad (9)$$

in *u (z, t)* in the quarter of the plane *z* > 0, *t* > 0. Let us specify which conditions should be given if *t* = 0 and if *z* = 0, so that the corresponding mixed boundary value problem has a unique solution.

It is known that the solution of the Cauchy problem for the telegraph equation in the half-plane *t* > 0 with conditions

$$u(z,0) = f(z), \quad \frac{\partial u}{\partial t}(z,0) = F(z) \qquad (10)$$

(see, for example, (Collin,1960)) has the form

$$u(z,t) = \frac{1}{2} f(z - at) - \frac{1}{2} \int_0^{z-at} \Phi(z,t,\xi)d\xi + \frac{1}{2} f(z + at) + \frac{1}{2} \int_0^{z+at} \Phi(z,t,\xi)d\xi \qquad (11)$$

,

where

$$\Phi(z,t,\xi) = \frac{1}{a} F(\xi) J_0\left(\frac{b}{a}\sqrt{a^2t^2 - (\xi - z)^2}\right) - btf(\xi) \frac{J_0'\left(\frac{b}{a}\sqrt{a^2t^2 - (\xi - z)^2}\right)}{\sqrt{a^2t^2 - (\xi - z)^2}},$$

and $J_0(z)$ is the Bessel function.

In formula (11), the first two terms are defined by the waves moving to the right with respect to axis *z*, and the second two terms are the waves moving to the left. If conditions (10) are given only on the half-line *z* > 0, then both groups of terms are defined in the lower infinite characteristic triangle *z* – *at* > 0.

By analogy, the Cauchy problem for the equation (is the same as (9))

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} + \frac{b^2}{a^2} u$$

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with boundary conditions

$$u(0,t)=g(t),\quad \frac{\partial u}{\partial z}(0,t)=G(t) \tag{12}$$

has the solution

$$u(z,t)=\frac{1}{2}g(t-z/a)-\frac{1}{2}\int_0^{t-z/a}\Psi(t,z,\xi)d\xi+\frac{1}{2}g(t+z/a)+\frac{1}{2}\int_0^{t+z/a}\Psi(t,z,\xi)d\xi, \tag{13}$$

where

$$\Psi(t,z,\xi)=aG(\xi)J_0\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)+b\frac{z}{a}g(\xi)\frac{J_0'\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)}{\sqrt{(\xi-t)^2-z^2/a^2}}.$$

Here the first group of terms gives the solutions moving to the right (about axis *z*), the second group gives the solutions moving to the left. Their sum is defined in the upper infinite characteristic triangle $z - at < 0$.

The left moving solutions from the two formulas should coincide. We equate their values on the characteristic $z = at = 0$. We obtain

$$\frac{1}{2}f(2at)+\frac{1}{2}\int_0^{2at}\Phi(at,t,\xi)d\xi=\frac{1}{2}g(2t)+\frac{1}{2}\int_0^{2t}\Psi(t,at,\xi)d\xi. \tag{14}$$

This equality establishes a link between boundary functions in the mixed boundary-value problem for the telegraph equation in the quarter of the plane, it is a solvability condition of the over-determined problem.

The unique solution of telegraph equation (9) in the quarter of the plane that satisfies the boundary conditions (10) and (12) exists if and only if (14) is satisfied.

Consequently, if two boundary conditions are given on the semi-axis $z > 0$, then it is sufficient to specify only one boundary condition on the semi-axis $t > 0$. The following assertion holds.

The solution of telegraph equation (9) in the quarter of the plane $z > 0$, $t > 0$, satisfying the boundary conditions

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$$u(z,0)=0, \quad \frac{\partial u}{\partial t}(z,0)=0, \quad u(0,t)=g(t), \tag{15}$$

has the form

$$u(z,t)=g(t-z/a)-b\frac{z}{a}\int_0^{t-z/a} g(\xi)\frac{J_0'\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)}{\sqrt{(\xi-t)^2-z^2/a^2}}d\xi. \tag{16}$$

The solution of telegraph equation (9) in the quarter of the plane z > 0, t > 0, satisfying the boundary conditions

$$u(z,0)=0, \quad \frac{\partial u}{\partial t}(z,0)=0, \quad \frac{\partial u}{\partial z}(0,t)=G(t), \tag{17}$$

has the form

$$u(z,t)=-a\int_0^{t-z/a} G(\xi)J_0\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)d\xi. \tag{18}$$

Proof. If the condition for the solvability of the overdetermined problem is satisfied, then it can be extended to the whole quarter of the plane and even to the half-plane $z = at > 0$, $z > 0$. Then for any z and $z + at > 0$

$$\frac{1}{2}f(z+at)+\frac{1}{2}\int_0^{z+at}\Phi(z,t,\xi)d\xi=\frac{1}{2}g(t+z/a)+\frac{1}{2}\int_0^{t+z/a}\Psi(t,z,\xi)d\xi.$$

For $f(x) = 0$, $F(x) = 0$ we have the equation

$$0=\frac{1}{2}g(t+z/a)+\frac{a}{2}\int_0^{t+z/a} G(\xi)J_0\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)d\xi$$
$$+\frac{bz}{2a}\int_0^{t+z/a} g(\xi)\frac{J_0'\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)}{\sqrt{(\xi-t)^2-z^2/a^2}}d\xi. \tag{19}$$

For $f(x) = 0$, $F(x) = 0$ in the lower characteristic triangle, the solution of the mixed problem is equal to zero. In the upper characteristic triangle we have

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$$u(z,t)=\frac{1}{2}g(t-z/a)-\frac{a}{2}\int_0^{t-z/a}G(\xi)J_0\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)d\xi$$
$$-\frac{bz}{2a}\int_0^{t-z/a}g(\xi)\frac{J_0'\left(b\sqrt{(\xi-t)^2-z^2/a^2}\right)}{\sqrt{(\xi-t)^2-z^2/a^2}}d\xi. \quad (20)$$

From the formulas (19) (we need to write *z* instead of -*z*) and (20), formulas (16) and (18) follow.

Formula (16) was obtained by another method in (Mittra,1971).

4. EXCITATION OF WAVES IN METAL WAVEGUIDE

If the characteristics of the electromagnetic field are known at the zero time and there are no other sources, then we can calculate the field for any values $t > 0$.

For example, if the values $E_z(x,y,z,0)=E_z^0(x,y,z)$ and $H_z(x,y,z,0)=H_z^0(x,y,z)$ for $t = 0$, then from the equations

$$\sum_m E_{z,m}(z,0)\chi_m\psi_m(x,y)=E_z^0(x,y,z),$$
$$\sum_m H_{z,m}(z,0)\lambda_m\varphi_m(x,y)=H_z^0(x,y,z)$$

we can obtain the coefficients of the Fourier series $E_{z,m}(z, t)$ and $H_{z,m}(z, t)$ for all m = 0,1,…. By the formulas giving solutions for the initial Cauchy problems for the corresponding telegraph equations, the functions $E_{z,m}(z, t)$ and $H_{z,m}(z, t)$ and, hence, all the components of the field are determined.

By sources on the cross-section of the waveguide, we can determine additional terms to the field components (assuming that the initial conditions are zero).

If for $z = 0$, the values of longitudinal components of the field $E_z(x, y, 0, t)$ and $H_z(x, y, 0, t)$ are given, then it is essential to obtain the coefficients of the expansions into Fourier series $E_{z,m}(0, t)$ and $H_{z,m}(0, t)$. The solutions of telegraph equations can be found from the formulas (16) and (18).

Now suppose that currents are given on the section of the waveguide, that is, the tangential components of the magnetic vector $H_x(x, y, 0, t)$ and $H_y(x, y, 0, t)$ are known (they are

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proportional to the components of the vector of surface current density). In this case, from the equations

$$-\varepsilon_0\varepsilon\sum_m \frac{\partial E_{z,m}}{\partial t}(0,t)\frac{\partial \psi_m}{\partial x}(x,y)+\sum_m \frac{\partial H_{z,m}}{\partial z}(0,t)\frac{\partial \varphi_m}{\partial y}(x,y)=H_x^0(x,y,t),$$

$$\varepsilon_0\varepsilon\sum_m \frac{\partial E_{z,m}}{\partial t}(0,t)\frac{\partial \psi_m}{\partial y}(x,y)+\sum_m \frac{\partial H_{z,m}}{\partial z}(0,t)\frac{\partial \varphi_m}{\partial x}(x,y)=H_y^0(x,y,t)$$

one determines

$$\varepsilon_0\varepsilon\chi_k \frac{\partial E_{z,k}}{\partial t}(0,t)=\iint_\Omega \left[\frac{\partial H_x^0}{\partial x}(x,y,t)-\frac{\partial H_y^0}{\partial y}(x,y,t)\right]\psi_k(x,y)dxdy,$$

$$-\lambda_k \frac{\partial H_{z,k}}{\partial t}(0,t)=\iint_\Omega \left[\frac{\partial H_x^0}{\partial y}(x,y,t)+\frac{\partial H_y^0}{\partial x}(x,y,t)\right]\varphi_k(x,y)dxdy.$$

The functions $\frac{\partial H_{z,m}}{\partial z}(0,t)$ are boundary functions of mixed problems (9), (17) with respect to $H_{z,m}(z, t)$. Since for classical solutions we have

$$E_{z,m}(0,t)=\int_0^t \frac{\partial E_{z,m}}{\partial t}(0,\tau)d\tau + E_{z,m}(0,0),$$

and for zero boundary conditions $E_{z,m}(0, 0) = 0$, then the boundary functions of mixed problems (9), (15) for the function $E_{z,m}$ are obtained.

5. NUMERICAL RESULTS

We conduct the numerical experiments for a rectangular waveguide with the cross-section $h_x \times h_y$. In the case of the rectangle, it is not difficult to find the eigenvalues

$$\lambda_m = \chi_m = \left(\frac{\pi m_x}{h_x}\right)^2 + \left(\frac{\pi m_y}{h_y}\right)^2$$

and the eigen functions

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$$\varphi_m(x,y) = cos\frac{\pi m_x x}{h_x} cos\frac{\pi m_y y}{h_y},$$

$$\psi_m(x,y) = sin\frac{\pi m_x x}{h_x} sin\frac{\pi m_y y}{h_y}.$$

We will assume that for the high mode ($m_x = m_y = 1$), a pulsed source of the following form is given:

$$H^0_{z,m}(t) = \{sin(ft), \quad 0 < t \le \pi/f; \quad 0, \quad t > \pi/f\}, \qquad (21)$$

where $f = 2.4 \times 10^8$.

Fig. 1 illustrates a graph of the dependence of the values of the function $H_{z,m}(z, t)$ on the spatial coordinate z and on the time t. Calculations are carried out by formula (18). It can be seen that the wave front moves with velocity at.

Note that the pulsed source does not remain a solitary wave, but it is distributed throughout the waveguide where the front passed. Without giving the estimates, we note that the values of $H_{z,m}(z, t)$ decrease in the direction from the wave front to the source as $t^{-1/2}$.

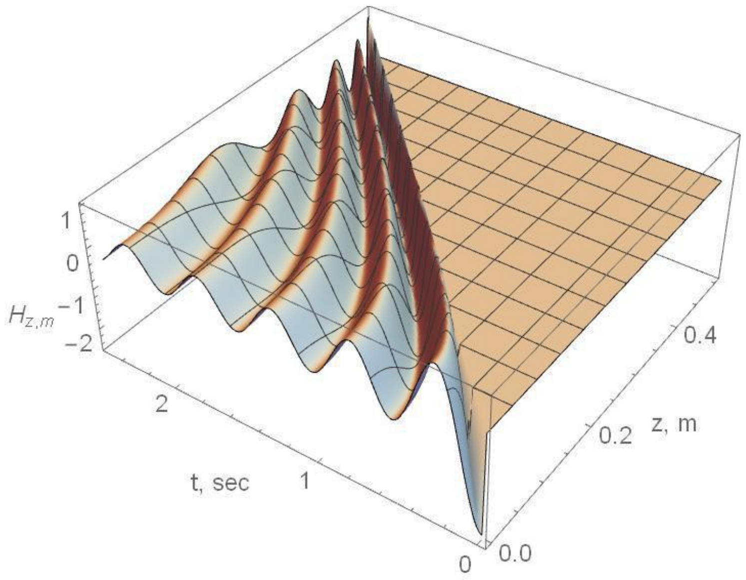

Figure 1. The distribution of the values of $H_{z,m}(z, t)$ for the highest mode of a rectangular waveguide with dimensions of 0.7 m at 0.7 m in coordinates z and t for a source of the form (21).

Now let the pulsed source be defined as follows:

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$$E^{0}_{z,m}(t)=\{sin(10t),\quad 0<t\le \pi/10;\quad 0,\quad t>\pi/10\}. \qquad (22)$$

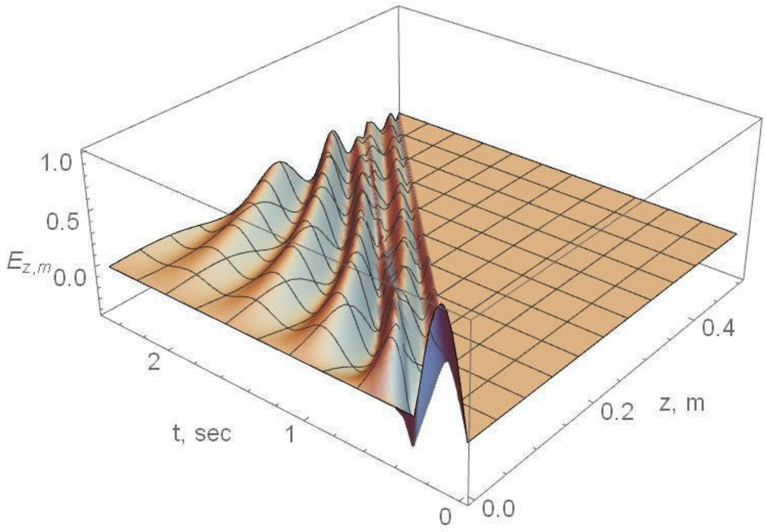

Figure 2. The distribution of the values of $E_{z,m}(z, t)$ for the highest mode of a rectangular waveguide with dimensions of 0.7 m at 0.7 m in coordinates z and t for a source of the form (22).

The graph of the dependence of the values of the function $E_{z,m}(z, t)$ on z and t, calculated from the formula (16), is shown in Fig. 2. As in the previous case, the pulse is distributed throughout the waveguide. However, the attenuation in t is stronger here, it is proportional to the value of $t^{-3/2}$.

6. CONCLUSIONS

It is shown that in the case of a non-harmonic time-dependent electromagnetic field, all the components of the natural waves of the waveguide with metal walls can be represented in the form of eigen function expansions of the two-dimensional Laplace operator. The coefficients of these expansions are expressed by the solutions of the telegraph equations.

It has been established that the unique solution of the telegraph equation in a quarter of the plane is determined by three conditions, by this, two conditions are the initial conditions, and one condition is the boundary condition.

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The problem of excitation of the electromagnetic field in an arbitrary cylindrical waveguide with metal walls by a source placed in the cross-section is reduced to an infinite set of mixed boundary problems for the telegraph equations, their solutions are written down in an explicit form.

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